

The Golay Code Outperforms the Extended Golay Code Under Hard-Decision Decoding

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Abstract

We show that the binary Golay code is slightly more power efficient than the extended binary Golay code under maximum-likelihood (ML), hard-decision decoding. In fact, if a codeword from the extended code is transmitted, one cannot achieve a higher probability of correct decoding than by simply ignoring the 24th symbol and using an ML decoder for the non-extended code on the first 23 symbols. This is so, despite the fact that using that last symbol would allow one to sometimes correct error patterns with weight four. To our knowledge the worse performance of the extended Golay code has not been previously noted, but it is noteworthy considering that it is the extended version of the code that has been preferred in many deployments.

1 Introduction

The many interesting properties of the Golay codes are well-studied [1, 2], and the codes have been deployed in several applications. The extended binary Golay code was used for NASA's Voyager mission during its encounters at Jupiter and Saturn [3]. It was also used to protect the data handling capabilities of NASA's Magellan mission to Venus [4], and the nonimaging science experiments of the Galileo mission [4]. Outside of NASA, the extended Golay code has been used as part of the Automatic Link Establishment protocol ITU-R F.1110 [5], it has been used in paging protocols [6], and it remains a standard for telemetry [7].

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One property that hasn't been reported in the literature, to the best of our knowledge, is that under maximum-likelihood (ML) decoding with hard-decisions, the extended binary Golay code, \mathcal{G}_{24} has slightly *worse* power efficiency than the binary Golay code, \mathcal{G}_{23} . This paper demonstrates this fact.

2 Performance of ML decoding with hard-decisions

In the following, we assume a binary symmetric channel, corresponding to a receiver that makes hard decisions. For the extended Golay code, the literature often describes an incomplete decoder capable of correcting three errors and detecting four errors. Such a decoder does not minimize the probability of codeword error, because it makes no attempt to guess at the correct codeword when four errors are detected. Since we desire to compare best-achievable error rate performance, we focus on codeword-error-rate-minimizing decoders, i.e., complete ML decoders that always output a closest codeword. The ML decoder offers no error detection, and we do not address error detection in this paper.

2.1 Binary Golay Code, \mathcal{G}_{23}

A complete ML decoder for the (23,12,7) Golay code will produce the correct codeword at its output whenever a hard-decision channel makes three or fewer errors in the 23 symbols. Thus, the codeword error rate (CWER) is

$$\text{CWER} = 1 - \sum_{i=0}^3 \binom{23}{i} p^i (1-p)^{23-i} \quad (1)$$

where p is the channel symbol error probability. This performance is shown in Figure 1 for a binary-input, additive white Gaussian noise (AWGN) channel with hard decision error probability

$$p = Q\left(\sqrt{\frac{2E_s}{N_0}}\right) = Q\left(\sqrt{\frac{2RE_b}{N_0}}\right) \quad (2)$$

where $R = 12/23$ is the code rate.

The bit error-rate (BER) performance is also shown in Figure 1. The BER is difficult to express analytically when the signal-to-noise ratio (SNR) is low, but the analysis becomes feasible at moderate to high SNR, because with high probability each codeword error results in seven code symbol errors. Each symbol (whether a systematic information-bearing symbol or a parity symbol) has 7/23 chance of being in error. Thus, at moderate and high SNR, the bit error rate is given by

$$\text{BER} \approx \frac{7}{23} \text{CWER} \quad (3)$$

which is tight (within about 0.1 dB) when $E_b/N_0 > 1$ dB.

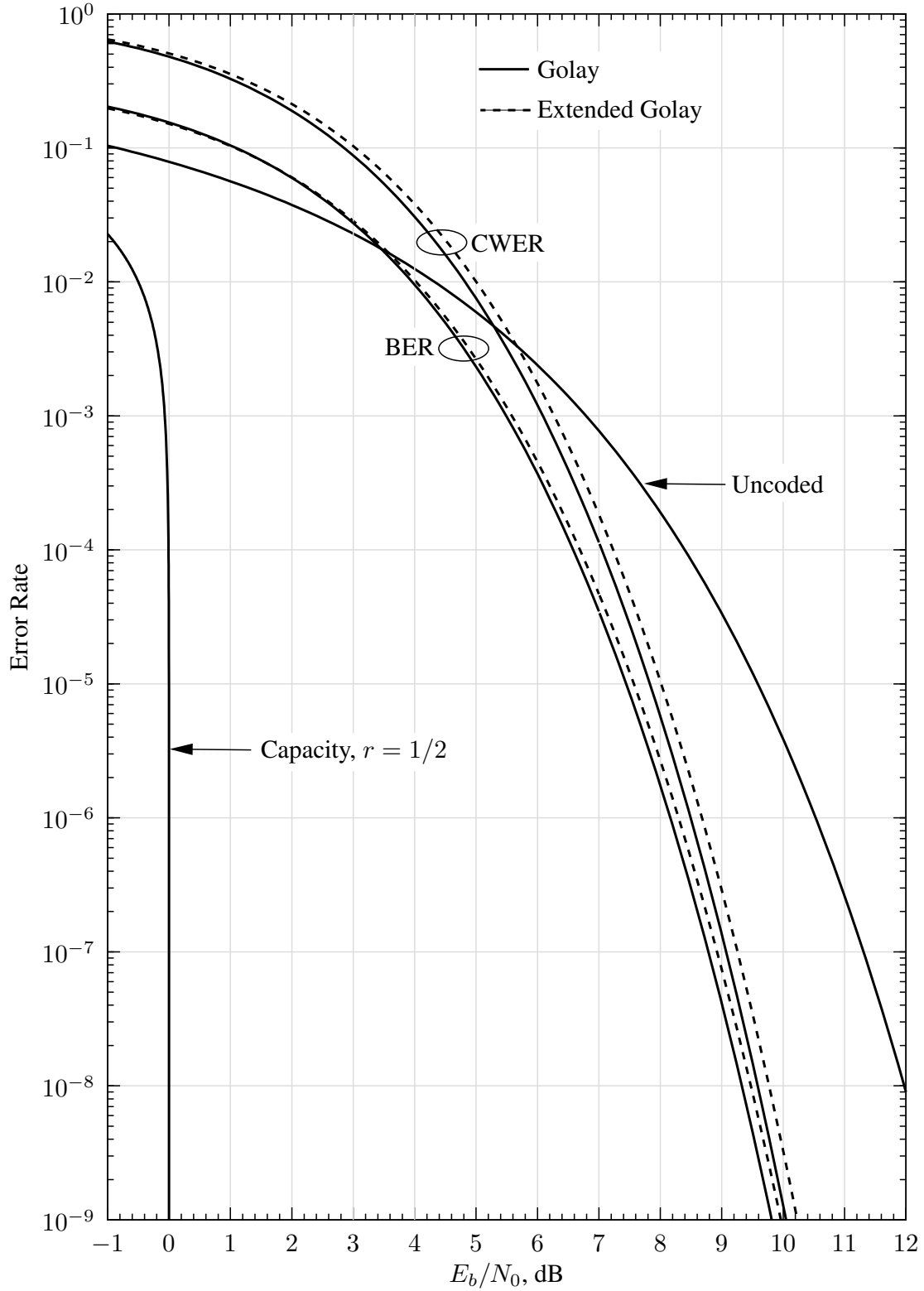


Figure 1: Performance of the binary Golay code and extended binary Golay code under hard-decision ML decoding, compared to capacity and uncoded transmission.

2.2 Extended Binary Golay Code, \mathcal{G}_{24}

Determining the performance of a complete ML decoder for \mathcal{G}_{24} is more involved because, unlike \mathcal{G}_{23} , the closest codeword to a received vector may not be unique. Determining decoder performance requires knowing how many codewords can be tied in this way, and how many bit errors are produced if the decoder guesses the wrong one. To help us, we start with two lemmas.

Lemma 1. *[8, 9] There is a unique codeword of \mathcal{G}_{24} of weight eight which has ones in any five given positions.*

Lemma 2. *There are five distinct codewords of \mathcal{G}_{24} of weight eight which have ones in any four given positions.*

Proof. Let $\mathbf{y} = (y_0, \dots, y_{23})$ be a vector with ones in four given positions. Let $\mathbf{y}' = \mathbf{y} + \mathbf{u}_i$, where i is one of the 20 indices for which $y_i = 0$, and where \mathbf{u}_i is a vector with a one in the i^{th} position. By Lemma 1, there is a unique codeword of \mathcal{G}_{24} of weight eight which has ones in the same five positions as \mathbf{y}' . Repeat this argument for each of the 20 values of i for which $y_i = 0$. This yields a list of 20 codewords. Each of these codewords occurs in the list four times, corresponding to the four positions that ones have been added to \mathbf{y} . Thus, there are $20/4 = 5$ distinct codewords of weight eight which have ones in the same four positions as \mathbf{y} . \square

We are now ready to state the distance properties we need to evaluate the performance of a complete ML decoder for \mathcal{G}_{24} .

Theorem 1. *For any vector $\mathbf{y} \in \{0, 1\}^{24}$, either*

- *There is a unique codeword $\mathbf{c} \in \mathcal{G}_{24}$ with $d(\mathbf{c}, \mathbf{y}) \leq 3$, or*
- *There are six distinct codewords $\mathbf{c} \in \mathcal{G}_{24}$ with $d(\mathbf{c}, \mathbf{y}) = 4$.*

Proof. If there is a codeword \mathbf{c} with $d(\mathbf{c}, \mathbf{y}) \leq 3$, then it must be unique, for if there were two codewords $\mathbf{c}^{(1)}$ and $\mathbf{c}^{(2)}$ of \mathcal{G}_{24} each within distance three of \mathbf{y} , then

$$d(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}) \leq d(\mathbf{c}^{(1)}, \mathbf{y}) + d(\mathbf{y}, \mathbf{c}^{(2)}) \leq 3 + 3 = 6. \quad (4)$$

Since \mathcal{G}_{24} has minimum distance eight, it must be that $\mathbf{c}^{(1)} = \mathbf{c}^{(2)}$.

Now suppose there is no codeword in \mathcal{G}_{24} within distance three of \mathbf{y} . There is a codeword of \mathcal{G}_{23} at distance at most three of (y_0, \dots, y_{22}) and adding a parity to this codeword produces a codeword of \mathcal{G}_{24} at distance at most four from \mathbf{y} . Since the distance is not three or less, it must be four.

We now determine the number of codewords of \mathcal{G}_{24} at distance four from \mathbf{y} . Since the code is linear, we lose no generality by assuming that one of the nearest codewords is the all-zero codeword, and thus, $w(\mathbf{y}) = 4$. By Lemma 2, there are five distinct codewords of weight eight and distance four from \mathbf{y} . Thus, together with the all-zero codeword, there are six codewords of \mathcal{G}_{24} which are distance four from \mathbf{y} . \square

To summarize, the ML decoder for \mathcal{G}_{24} will find a unique codeword if there is a codeword within distance three from the received vector, and otherwise it will output one of the six codewords at distance four (and it will have a 1/6 chance of being correct in that case). Thus, the codeword error rate is given by

$$\text{CWER} = 1 - \frac{1}{6} \binom{24}{4} p^4 (1-p)^{20} - \sum_{i=0}^3 \binom{24}{i} p^i (1-p)^{24-i} \quad (5)$$

This performance is shown in Figure 1. At moderate to high SNR, when a codeword error is made then with high probability it results in exactly eight code symbol errors. So on average, the code symbols have $8/24 = 1/3$ chance of being in error. When the decoder detects four errors, if it randomly selects from among the six codewords at distance four, this 1/3 average applies equally to the systematic and parity bits, in which case at moderate and high SNR the bit error rate would be

$$\text{BER} \approx \frac{1}{3} \times \text{CWER} \quad (6)$$

But the decoder need not randomly select from among the six codewords at distance four. Instead, the decoder may select a codeword at distance four whose systematic bits agree with the systematic bits of the received vector in the most number of positions. This does not affect the CWER, but when a codeword error is made, there are fewer systematic bit errors than parity symbol errors, on average. In fact, a simulation indicates that only about 3.1 of the twelve systematic bits are in error per codeword in error (instead of four in twelve for the decoder which randomly selects the codeword at distance four), or

$$\text{BER} \approx 0.26 \times \text{CWER} \quad (7)$$

The performance of this decoder is shown in Figure 1. At $\text{BER} = 10^{-6}$, \mathcal{G}_{24} has a coding gain of 2.1 dB, and a gap of 8.4 dB to the capacity of rate 1/2 coding on an unconstrained-input channel.

The analysis above assumes the decoder is required to output a codeword. If all one cares about is BER, and not producing a valid decoded codeword, the BER can be improved further. When four errors are detected, one can simply output the received systematic bits exactly as they were received from the channel. This is unlikely to be fully correct, but on average these bits contain only half of the four errors, or 2 bit errors per codeword. This compares favorably to the decoder above, which when faced with 4 channel errors decodes to the correct codeword 1/6 of the time and the other 5/6 of the time produces on average 3.1 bit errors, or about 2.6 bit errors per codeword.

2.3 Comparison of \mathcal{G}_{23} and \mathcal{G}_{24} Performance

One might expect that, because the minimum distance of \mathcal{G}_{24} is higher than that of \mathcal{G}_{23} (8 vs. 7), it is a better code that is more efficient at correcting errors. Remarkably, when the channel makes hard decisions, this is not the case, even at high SNR! Figure 1 shows that \mathcal{G}_{23} performs better than \mathcal{G}_{24} by about 0.2 dB, but it is helpful to explain why. The reason is that transmitting the parity

bit uses slightly more energy than is saved by being able to correct 1/6 of the weight-four error patterns.

To see this, suppose codewords of \mathcal{G}_{24} are transmitted on a binary symmetric channel with cross-over probability p . We compare two decoders:

- Decoder D_{23} is a complete ML decoder for \mathcal{G}_{23} , and ignores the 24th symbol.
- Decoder D_{24} is a complete ML decoder for \mathcal{G}_{24} .

Which decoder has a better CWER? We can answer this by comparing two quantities.

1. *Error patterns that decoder D_{24} corrects that decoder D_{23} does not.*

If the channel makes four errors in the first 23 symbols and the 24th symbol is received correctly, then D_{23} will decode in error, and with probability 1/6 D_{24} will decode correctly. Thus, the probability D_{24} is correct and D_{23} is not, is

$$\frac{1}{6} \binom{23}{4} p^4 (1-p)^{20} \quad (8)$$

2. *Error patterns that decoder D_{23} corrects that decoder D_{24} does not.*

If the channel makes three errors in the first 23 symbols and the 24th symbol is also received in error, decoder D_{23} will find the correct answer, and with probability 5/6 decoder D_{24} will not find the correct answer. Thus, the probability D_{23} is correct and D_{24} is not, is

$$\frac{5}{6} \binom{23}{3} p^4 (1-p)^{20} \quad (9)$$

In all other situations, either both decoders find the correct codeword, or both decoders produce a codeword error. Since

$$5 \binom{23}{3} = \binom{23}{4} = 8855 \quad (10)$$

it follows that for any given p , decoder D_{23} and D_{24} have the identical CWER! Thus, if a codeword from \mathcal{G}_{24} is transmitted, we cannot do better than simply ignoring the 24th symbol and using the complete ML decoder for \mathcal{G}_{23} on the first 23 symbols. This is so, despite the fact that using that last symbol would allow us to sometimes correct error patterns with weight four—that advantage is exactly balanced by the chance that the last symbol will be received in error and prevent proper decoding.

The story is more nuanced if we compare the BER. By comparing (3) to (6), we see that for any given p , the random-ML-codeword decoder for \mathcal{G}_{24} is *worse* than the ML decoder for \mathcal{G}_{23} by a factor of $(1/3)/(7/23) = 23/21$. On the other hand, for a given value of p , the carefully designed BER-minimizing decoder for D_{24} discussed above (see (7)) has a lower BER than that of decoder D_{23} , since $0.26 < 7/23$.

This analysis addresses the question of what to do if the 24th symbol has been transmitted. Since the CWER is the same whether we make use of it or not, we are even better off by not transmitting it at all. This allows a savings in energy of $10 \log_{10} 24/23 \approx 0.18$ dB. This is why the hard-decision CWER curves in Figure 1 are separated by exactly this amount at all SNRs. The BER curves, as expected, are slightly closer together, but \mathcal{G}_{23} is still seen to be better than \mathcal{G}_{24} by about 0.13 dB.

We take care to note that \mathcal{G}_{23} outperforms \mathcal{G}_{24} only on a *hard-decision* channel. Under ML soft-decision decoding, \mathcal{G}_{24} outperforms \mathcal{G}_{23} , which can be verified by simulation or, at high SNR, by comparing the union bound expressions for the two codes.

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